

# PROJECTIONS OF ORBITAL MEASURES, GELFAND–TSETLIN POLYTOPES, AND SPLINES

GRIGORI OLSHANSKI

**ABSTRACT.** The unitary group  $U(N)$  acts by conjugations on the space  $\mathcal{H}(N)$  of  $N \times N$  Hermitian matrices, and every orbit of this action carries a unique invariant probability measure called an orbital measure. Consider the projection of the space  $\mathcal{H}(N)$  onto the real line assigning to an Hermitian matrix its  $(1, 1)$ -entry. Under this projection, the density of the pushforward of a generic orbital measure is a spline function with  $N$  knots. This fact was pointed out by Andrei Okounkov in 1996, and the goal of the paper is to propose a multidimensional generalization. Namely, it turns out that if instead of the  $(1, 1)$ -entry we cut out the upper left matrix corner of arbitrary size  $K \times K$ , where  $K = 2, \dots, N - 1$ , then the pushforward of a generic orbital measure is still computable: its density is given by a  $K \times K$  determinant composed from one-dimensional splines. The result can also be reformulated in terms of projections of the Gelfand–Tsetlin polytopes.

## CONTENTS

1. Introduction	1
2. Preliminaries	3
3. Projections of orbital measures	5
4. Acknowledgement	11
References	11

## 1. INTRODUCTION

**Orbital measures.** Let  $\mathcal{H}(N)$  be the space of  $N \times N$  Hermitian matrices. For  $K = 1, \dots, N - 1$ , we denote by  $p_K^N : \mathcal{H}(N) \rightarrow \mathcal{H}(K)$  the linear projection consisting in deleting from the matrix  $H \in \mathcal{H}(N)$  its last  $N - K$  rows and columns. We call  $p_K^N(H)$ , the image of  $H$  under this projection, the  $K \times K$  *corner* of  $H$ .

The unitary group  $U(N)$  acts on  $\mathcal{H}(N)$  by conjugations, and because  $U(N)$  is compact, each orbit of this action carries a unique invariant probability measure, which we call the *orbital measure*. Given an orbital measure  $\mu$  on  $\mathcal{H}(N)$ , denote by  $p_K^N(\mu)$  its pushforward under projection  $p_K^N$ . Our goal is to describe  $p_K^N(\mu)$ .

The orbits in  $\mathcal{H}(N)$  (and hence the orbital measures) can be indexed by  $N$ -tuples of weakly increasing real numbers  $X = (x_1 \leq \dots \leq x_N)$ , the matrix eigenvalues.

Let  $\mathcal{X}(N) \subset \mathbb{R}^N$  denote the set of all such  $X$ 's. Given  $X \in \mathcal{X}(N)$ , we write  $O_X$  and  $\mu_X$  for the corresponding orbit and orbital measure, respectively.

Since  $p_K^N(\mu_X)$  is a  $U(K)$ -invariant probability measure on  $\mathcal{H}(K)$ , it can be uniquely decomposed into a continual convex combination of orbital measures, governed by a probability measure  $\nu_{X,K}$  on the parameter space  $\mathcal{X}(K)$ . That is,  $\nu_{X,K}$  is characterized by the property that, for an arbitrary Borel subset  $S \subseteq \mathcal{X}(N)$ ,

$$(p_K^N(\mu_X))(S) = \int_{Y \in \mathcal{X}(K)} \mu_Y(S) \nu_{X,K}(dY).$$

The measure  $\nu_{X,K}$  can be called the *radial part* of measure  $p_K^N(\mu_X)$ .

**Main result.** Denote by  $\mathcal{X}^0(N)$  the interior of  $\mathcal{X}(N)$ ; that is,  $\mathcal{X}^0(N)$  consists of  $N$ -tuples of *strictly* increasing real numbers. If  $X \in \mathcal{X}^0(N)$ , then  $\nu_{X,K}$  is absolutely continuous with respect to Lebesgue measure on  $\mathcal{X}(K) \subset \mathbb{R}^K$ , and the main result, Theorem 3.3, gives an explicit formula for the density of  $\nu_{X,K}$ .

In the case  $K = 1$  the target space of the projection is the real line, and the density in question coincides with a *B-spline*, a certain piecewise polynomial function on  $\mathbb{R}$  (this fact was observed by Andrei Okounkov). In the general case, it turns out that the density of  $\nu_{X,K}$  is expressed through a  $K \times K$  determinant composed from some B-splines.

As the reader will see, the proof of Theorem 3.3 is straightforward and elementary. The main reason why I believe the result may be of interest is the very appearance of splines, which are objects of classical and numerical analysis, in a problem of representation-theoretic origin.

**Gelfand–Tsetlin polytopes.** Before explaining a connection with representation theory I want to give a different interpretation of the measure  $\nu_{X,K}$ .

For  $X \in \mathcal{X}(N)$  and  $Y \in \mathcal{X}(N-1)$ , write  $Y \prec X$  or  $X \succ Y$  if the coordinates of  $X$  and  $Y$  *interlace*, that is

$$x_1 \leq y_1 \leq x_2 \leq \cdots \leq x_{N-1} \leq y_{N-1} \leq x_N.$$

Given  $X \in \mathcal{X}(N)$ , the corresponding *Gelfand–Tsetlin polytope*  $P_X$  is the compact convex subset in the vector space

$$\mathbb{R}^{N-1} \times \mathbb{R}^{N-2} \times \cdots \times \mathbb{R} = \mathbb{R}^{N(N-1)/2},$$

formed by triangular arrays subject to the interlacement constraints:

$$P_X := \{(Y^{(N-1)}, \dots, Y^{(1)}) \in \mathbb{R}^{N(N-1)/2} : X \succ Y^{(N-1)} \succ \cdots \succ Y^{(1)}\}.$$

Consider the map assigning to a matrix  $H \in O_X$  the array formed by the collections of eigenvalues of its corners  $p_{N-1}^N(H), p_{N-2}^N(H), \dots, p_1^N(H)$ . It is well known (see Corollary 3.2 below) that this map projects the orbit  $O_X$  onto the polytope  $P_X$  and takes  $\mu_X$  to the uniform measure on  $P_X$  (that is, the normalized Lebesgue measure). Next, given  $K = 1, \dots, N-1$ , consider the natural projection  $P_X \rightarrow \mathcal{X}(K)$  extracting from the array  $(Y^{(N-1)}, \dots, Y^{(1)})$  its  $K$ th component  $Y^{(K)}$ . The

measure  $\nu_{X,K}$  is nothing else than the pushforward of the uniform measure under the latter projection.

**Discrete version of the problem: relative dimension in Gelfand–Tsetlin graph.** Let  $\mathbb{GT}_N := \mathcal{X}(N) \cap \mathbb{Z}^N$  be the set of weakly increasing  $N$ -tuples of integers. The elements of  $\mathbb{GT}_N$  are in bijection with the irreducible representations of the group  $U(N)$ : with  $X = (x_1, \dots, x_N) \in \mathbb{GT}_N$  we associate the irreducible representation  $T_X$  with signature (=highest weight)  $\hat{X} := (x_N, \dots, x_1)$ . Here we pass from  $X$  to  $\hat{X}$ , because the coordinates of signatures are usually written in the descending order, see Weyl [13].

Let  $X \in \mathbb{GT}_N$  and consider the finite set  $P_X^{\mathbb{Z}} := P_X \cap \mathbb{Z}^{N(N-1)/2}$  consisting of integral points in the polytope  $P_X$ . Let us replace the uniform measure on  $P_X$  by the uniform measure on  $P_X^{\mathbb{Z}}$  (that is, the normalized counting measure). Next, given  $K = 1, \dots, N-1$ , we consider again the same projection  $P_X \rightarrow \mathcal{X}(K)$  as before and denote by  $\nu_{X,K}^{\mathbb{Z}}$  the pushforward of the uniform measure on  $P_X^{\mathbb{Z}}$ . Evidently,  $\nu_{X,K}^{\mathbb{Z}}$  is a probability measure with finite support.

Elements of  $P_X^{\mathbb{Z}}$  are the *Gelfand–Tsetlin schemes* (also called Gelfand–Tsetlin patterns) with top row  $X$ ; they parameterize the elements of Gelfand–Tsetlin basis in  $T_X$ . By the very definition of  $\nu_{X,K}^{\mathbb{Z}}$ , for  $Y \in \mathbb{GT}_K$ , the quantity  $\nu_{X,K}^{\mathbb{Z}}(Y)$  (the mass assigning by  $\nu_{X,K}^{\mathbb{Z}}$  to  $Y$ ) equals the fraction of the schemes with the  $K$ th row equal to  $Y$ . This quantity is the same as the relative dimension of the isotypic component of  $T_Y$  in the restriction of  $T_X$  to the subgroup  $U(K) \subset U(N)$ .

The *Gelfand–Tsetlin graph* has the vertex set  $\mathbb{GT}_1 \sqcup \mathbb{GT}_2 \sqcup \dots$  and the edges formed by couples  $Y \prec X$ . In the terminology of Borodin–Olshanski [3],  $\nu_{X,K}^{\mathbb{Z}}(Y)$  is the *relative dimension* of the vertex  $Y \in \mathbb{GT}_K$  with respect to the vertex  $X \in \mathbb{GT}_N$ . In [3], we derived a determinantal formula for the relative dimension (see also Petrov [11] for a different proof). That formula can be viewed as a discrete version of the formula of Theorem 3.3.

I first guessed the formula of Theorem 3.3 by degenerating the “discrete” formula of [3]. However, this is not an optimal way of derivation, because the discrete case is much more difficult than the continuous one. I am grateful to Alexei Borodin for the suggestion to study the degeneration of the “discrete” formula. Note that from the comparison of the measures  $\nu_{N,K}$  and  $\nu_{X,K}^{\mathbb{Z}}$  it is seen that the former should be related to the latter by a scaling limit transition.

## 2. PRELIMINARIES

The *fundamental spline* with  $n \geq 2$  knots  $y_1 < \dots < y_n$  can be characterized as the only function  $a \mapsto M(a; y_1, \dots, y_n)$  on  $\mathbb{R}$  of class  $C^{n-3}$ , vanishing outside the interval  $(y_1, y_n)$ , equal to a polynomial of degree  $\leq n-2$  on each interval  $(y_i, y_{i+1})$ ,

and normalized by the condition

$$\int_{-\infty}^{+\infty} M(a; y_1, \dots, y_n) da = 1.$$

Here is an explicit expression:

$$M(a; y_1, \dots, y_n) := (n-1) \sum_{i: y_i > a} \frac{(y_i - a)^{n-2}}{\prod_{r: r \neq i} (y_i - y_r)}. \quad (1)$$

In particular, for  $n = 2$

$$M(a; y_1, y_2) = \frac{\mathbf{1}_{y_1 \leq a \leq y_2}}{y_2 - y_1}.$$

**Remark 2.1.** The above definition is taken from Curry–Schoenberg [4]. In the subsequent publications, Schoenberg changed the term to *B-spline*. The latter term became commonly used. However, in the modern literature, it more often refers to the function

$$B(a; y_1, \dots, y_n) := (y_n - y_1) \sum_{i: y_i > a} \frac{(y_i - a)^{n-2}}{\prod_{r: r \neq i} (y_i - y_r)}, \quad (2)$$

which differs from  $M(a; y_1, \dots, y_n)$  by the numerical factor  $(y_n - y_1)/(n-1)$ ; see, e.g., de Boor [2] or Phillips [12]. The normalization in (2) has its own advantages, but we will not use it. Note also that  $M(a; y_1, \dots, y_n)$  is a special case of *Peano kernel*, see Davis [5], Faraut [7].

We need two well-known formulas relating  $M(a; y_1, \dots, y_n)$  to divided differences (see, e.g., [4], [7]).

Recall that the *divided difference* of a function  $f(x)$  on points  $y_1, \dots, y_n$  is defined recursively by

$$f[y_1, y_2] = \frac{f(y_2) - f(y_1)}{y_2 - y_1}, \quad f[y_1, y_2, y_3] = \frac{f[y_2, y_3] - f[y_1, y_2]}{y_3 - y_1},$$

and so on; the final step is

$$f[y_1, \dots, y_n] = \frac{f[y_2, \dots, y_n] - f[y_1, \dots, y_{n-1}]}{y_n - y_1}. \quad (3)$$

Next, set

$$x_+^s = \begin{cases} x^s, & x > 0 \\ 0, & x \leq 0. \end{cases}$$

In this notation, the first formula in question is

$$M(a; y_1, \dots, y_n) = (n-1)f[y_1, \dots, y_n], \quad \text{where } f(x) := (x - a)_+^{n-2}, \quad (4)$$

and the second formula is

$$f[y_1, \dots, y_n] = \frac{1}{(n-1)!} \int M(a; y_1, \dots, y_n) f^{(n-1)}(a) da. \quad (5)$$

In (5),  $f$  is assumed being a function on  $\mathbb{R}$  with piecewise continuous derivative of order  $n-1$ . In particular, (5) is applicable to  $f(x) = (x-t)_+^{n-1}$ , which is used in the lemma below.

To shorten the notation, let us abbreviate  $Y := (y_1 < \dots < y_n)$ .

**Lemma 2.2.** *Fix  $n = 2, 3, \dots$  and an  $n$ -tuple  $Y = (y_1 < \dots < y_n) \in \mathcal{X}^0(N)$ . For an arbitrary  $b \in \mathbb{R}$  set*

$$f_b(x) := (x-b)_+^{n-1}, \quad x \in \mathbb{R}.$$

One has

$$\int_{-\infty}^c M(a; Y) da = 1 - f_c[Y], \quad c \in \mathbb{R}, \quad (6)$$

$$\int_b^c M(a; Y) da = f_b[Y] - f_c[Y], \quad b < c, \quad (7)$$

$$\int_b^{+\infty} M(a; Y) da = f_b[Y], \quad b \in \mathbb{R}. \quad (8)$$

*Proof.* To check (7), we apply (5) to  $f(x) = f_b(x) - f_c(x)$ , which is justified, see the comment just after (5). Then in the left-hand side of (5) we get  $f_b[Y] - f_c[Y]$ . Next, observe that the  $(n-1)$ th derivative of  $f_t(x)$  equals  $(n-1)! \mathbf{1}_{x \geq b}$ , so that

$$f^{(n-1)}(a) = (n-1)! (\mathbf{1}_{a \geq b} - \mathbf{1}_{a \geq c}) = (n-1)! \mathbf{1}_{b \leq a < c}.$$

Therefore, in the right-hand side we get  $\int_b^c M(a; Y) da$ , which proves (7).

Now (8) follows from (7) by setting  $c = +\infty$ , and (6) follows from (8), because the total integral of the  $M(a; Y)$  equals 1.  $\square$

### 3. PROJECTIONS OF ORBITAL MEASURES

We keep to the notation of Sections 1 and 2

Given  $X \in \mathcal{X}(N)$ , the pushforward of the orbital measure  $\mu_X$  under the map

$$O_X \ni H \mapsto \text{the spectrum of } p_{N-1}^N(H)$$

can be viewed as a probability measure on  $\mathcal{X}(N-1)$  depending on  $X$  as a parameter; let us denote it by  $\Lambda_{N-1}^N(X, \cdot)$  or  $\Lambda_{N-1}^N(X, dY)$ . We regard  $\Lambda(X, dY)$  as a Markov kernel.

By classical Rayleigh's theorem, the eigenvalues of a matrix  $H \in \mathcal{X}(N)$  and its corner  $p_{N-1}^N(H)$  interlace. Therefore, the measure  $\Lambda_{N-1}^N(X, \cdot)$  is concentrated on the subset

$$\{Y \in \mathcal{X}(N-1) : Y \prec X\} \subset R^{N-1}. \quad (9)$$

**Proposition 3.1.** *Assume  $X = (x_1, \dots, x_N) \in \mathcal{X}^0(N)$ . Then the measure  $\Lambda_{N-1}^N(X, \cdot)$  is absolutely continuous with respect to Lebesgue measure on the set (9), and the density of  $\Lambda_{N-1}^N(X, \cdot)$ , denoted by  $\Lambda_{N-1}^N(X, Y)$ , is given by*

$$\Lambda_{N-1}^N(X, Y) = (N-1)! \frac{V(Y)}{V(X)} \mathbf{1}_{Y \prec X}, \quad (10)$$

where we use the notation

$$V(X) = \prod_{j>i} (x_j - x_i)$$

and the symbol  $\mathbf{1}_{Y \prec X}$  equals 1 or 0 depending on whether  $Y \prec X$  or not.

*Proof.* To the best of my knowledge, a published proof first appeared in Baryshnikov [1, Proposition 4.2]. However, the argument given in [1] was known earlier: it is hidden in the first computation of the spherical functions of the groups  $SL(N, \mathbb{C})$  due to Gelfand and Naimark, see [8, §9]. Note also that a more general result can be found in Neretin [9].

Here is a different proof. Consider the Laplace transform of the orbital measure  $\mu_X$ :

$$\widehat{\mu}_X(Z) := \int e^{\text{Tr}(ZH)} \mu_X(dH), \quad (11)$$

where  $Z$  is a complex  $N \times N$  matrix. The integral in the right-hand side is often called the *Harish-Chandra–Itzykson–Zuber integral*. Its value is given by a well-known formula (see, e.g., Olshanski–Vershik [10, Corollary 5.2]):

$$\widehat{\mu}_X(Z) = c_N \frac{\det[e^{z_i x_j}]_{i,j=1}^N}{\prod_{j>i} (z_j - z_i)(x_j - x_i)}, \quad (12)$$

where  $z_1, \dots, z_N$  are the eigenvalues of  $Z$  and

$$c_N = (N-1)!(N-2)! \dots 0!$$

(note that the right-hand side of (12) does not depend on the enumeration of the eigenvalues of  $Z$ ).

The claim of the proposition is equivalent to the following equality: Assume that the entries in the last row and column of  $Z$  equal 0, so that  $Z$  has the form

$$Z = \begin{bmatrix} \widetilde{Z} & 0 \\ 0 & 0 \end{bmatrix}, \quad (13)$$

where  $\widetilde{Z}$  is a complex matrix of size  $(N-1) \times (N-1)$ ; then

$$\widehat{\mu}_X(Z) = \frac{(N-1)!}{V(X)} \int_{Y \prec X} V(Y) \widehat{\mu}_Y(\widetilde{Z}) dY. \quad (14)$$

To prove (14), consider the matrix  $T := [e^{z_i x_j}]$  in the right-hand side of (12). Since  $Z$  has the form (13), at least one of the eigenvalues  $z_1, \dots, z_N$  equals 0. It is

convenient to slightly change the enumeration and denote the eigenvalues as  $z_0 = 0, z_1, \dots, z_{N-1}$ . In accordance to this we will assume that the row number  $i$  of  $T$  ranges over  $\{0, \dots, N-1\}$  while the column index  $j$  ranges over  $\{1, \dots, N\}$ . Since  $z_0 = 0$ , the 0th row of  $T$  is  $(1, \dots, 1)$ . Let us subtract the  $(N-1)$ th column from the  $N$ th one, then subtract the  $(N-2)$ th column from the  $(N-1)$ th one, etc. This gives  $\det T = \det \tilde{T}$ , where  $\tilde{T}$  stands for the matrix of order  $N-1$  with the entries

$$\tilde{T}_{i,j} = e^{z_i x_{j+1}} - e^{z_i x_j} = z_i \int_{x_i}^{x_j} e^{z_i y_j} dy_j, \quad i, j = 1, \dots, N-1.$$

It follows

$$\det \tilde{T} = z_1 \dots z_N \int_{Y \prec X} dY \det[e^{z_i y_j}]_{i,j=1}^{N-1},$$

so that

$$\hat{\mu}_X(Z) = c_N \frac{z_1 \dots z_N \int_{Y \prec X} dY \det[e^{z_i y_j}]_{i,j=1}^{N-1}}{\prod_{N-1 \geq j > i \geq 0} (z_j - z_i) \cdot V(X)}.$$

Next, because  $z_0 = 0$ , the product over  $j > i$  in the denominator equals

$$z_1 \dots z_N \prod_{N-1 \geq j > i \geq 1} (z_j - z_i),$$

so that the product  $z_1 \dots z_N$  is cancelled out. Taking into account the fact that  $\hat{\mu}_Y(\tilde{Z})$  is given by the determinantal formula similar to (12) and using the obvious relation  $c_N = (N-1)!c_{N-1}$  we finally get the desired equality (14).  $\square$

From Proposition 3.1 it is easy to deduce the following corollary (see also [1, Proposition 4.7] and Defosseux [6]).

**Corollary 3.2.** *Fix  $X \in \mathcal{X}^0(N)$  and let  $H$  range over  $O_X$ . The map assigning to  $H$  the collection of the eigenvalues of the corners  $p_K^N(H)$ , where  $K = N-1, N-2, \dots, 1$ , projects  $O_X$  onto the Gelfand–Tsetlin polytope  $P_X$  and takes the measure  $\mu_X$  to the Lebesgue measure multiplied by the constant*

$$\frac{(N-1)!(N-2)! \dots 0!}{V(X)}.$$

*In particular, the volume of  $P_X$  in the natural coordinates is equal to the inverse of the above quantity.*

Recall that  $\nu_{X,K}$  stands for the radial part of the  $K \times K$  corner of the random matrix  $H \in O_X$ , driven by the orbital measure  $\mu_X$  (see Section 1), and  $M(a; y_1, \dots, y_n)$  denotes the fundamental spline with  $n$  knots  $y_1, \dots, y_n$  (see (1) and (4)).

**Theorem 3.3.** *Fix  $X = (x_1, \dots, x_N) \in \mathcal{X}^0(N)$ . For any  $K = 1, \dots, N-1$ , the measure  $\nu_{X,K}$  on  $\mathcal{X}(K)$  is absolutely continuous with respect to Lebesgue measure and has the density*

$$M(a_1, \dots, a_K; x_1, \dots, x_N) := c_{N,K} \frac{V(A) \det [M(a_j; x_i, \dots, x_{N-K+i})]_{i,j=1}^K}{\prod_{(j,i): j-i \geq N-K+1} (x_j - x_i)}, \quad (15)$$

where

$$c_{N,K} = \prod_{i=1}^{K-1} \binom{N-K+i}{i}.$$

Note that for  $K = 1$  the right-hand side reduces to the fundamental spline with knots  $x_1, \dots, x_N$ . Thus, in the case  $K = 1$  the theorem says that the density of the measure  $\nu_{N,1}$  on  $\mathbb{R}$  coincides with the spline  $M(a; x_1, \dots, x_N)$ . This simple but important claim is due to Andrei Okounkov, see [10, Proposition 8.2].

*Proof.* We argue by induction on  $K$ , starting with  $K = N-1$  and ending at  $K = 1$ .

*Step 1.* Examine the case  $K = N-1$ , which is the base of induction. We have  $\nu_{X,N-1}(dA) = \Lambda_{N-1}^N(X, dA)$ . By proposition 3.1, the measure  $\Lambda_{N-1}^N(X, \cdot)$  on  $\mathcal{X}(N-1)$  is absolutely continuous with respect to Lebesgue measure and has density  $\Lambda_{N-1}^N(X, A)$  given by (10). Thus, we have to check that  $\Lambda_{N-1}^N(X, A)$  coincides with the quantity  $M(a_1, \dots, a_{N-1}; x_1, \dots, x_N)$  given by the right-hand side of (15), where we have to take  $K = N-1$ . That is, the desired equality has the form

$$(N-1)! \frac{V(A)}{V(X)} \mathbf{1}_{A \prec X} = c_{N,N-1} \frac{V(A) \det [M(a_j; x_i, x_{i+1})]_{i,j=1}^{N-1}}{\prod_{(j,i): j-i \geq 2} (x_j - x_i)}.$$

Since  $c_{N,N-1} = (N-1)!$ , the desired equality reduces to

$$\det [M(a_j; x_i, x_{i+1})]_{i,j=1}^{N-1} = \frac{\mathbf{1}_{A \prec X}}{(x_2 - x_1)(x_3 - x_2) \dots (x_N - x_{N-1})}.$$

Observe that the  $(i, j)$ -entry in the determinant is the quantity

$$M(a_j; x_i, x_{i+1}) = \frac{\mathbf{1}_{x_i \leq a_j \leq x_{i+1}}}{x_{i+1} - x_i},$$

which vanishes unless  $a_j \in [x_i, x_{i+1}]$ . Since  $a_1 \leq \dots \leq a_{N-1}$ , the determinant vanishes unless  $A \prec X$ . Furthermore, if  $A \prec X$ , then the matrix under the sign of determinant is diagonal, so the determinant equals the product of the diagonal entries, which equals

$$\frac{1}{(x_2 - x_1)(x_3 - x_2) \dots (x_N - x_{N-1})},$$

as required.



*Step 2.* Given  $K = 1, \dots, N-1$ , we consider the superposition of Markov kernels

$$\Lambda_K^N := \Lambda_{N-1}^N \Lambda_{N-2}^{N-1} \dots \Lambda_K^{K+1}.$$

In more detail, the result is a Markov kernel on  $\mathcal{X}(N) \times \mathcal{X}(K)$  given by

$$\Lambda_K^N(X, dA) = \int \Lambda_{N-1}^N(X, dY^{(N-1)}) \Lambda_{N-2}^{N-1}(Y^{(N-1)}, dY^{(N-2)}) \dots \Lambda_K^{K+1}(Y^{(K+1)}, dA),$$

where the integral is taken over variables  $Y^{(N-1)}, \dots, Y^{(K+1)}$ . Obviously,  $\Lambda_K^N(X, dA) = \nu_{X,K}(dA)$ , which entails the recurrence relation

$$\nu_{X,K-1} = \nu_{X,K} \Lambda_{K-1}^K, \quad K = N-1, N-2, \dots, 2, \quad (16)$$

where, by definition,  $\nu_{X,K} \Lambda_{K-1}^K$  is the measure on  $\mathcal{X}(K-1)$  given by

$$(\nu_{X,K} \Lambda_{K-1}^K)(dB) = \int_{A \in \mathcal{X}(K)} \nu_{X,K}(dA) \Lambda_{K-1}^K(A, dB). \quad (17)$$

*Step 3.* Assume now that the claim of the theorem holds for some  $K \geq 2$  and deduce from this that it also holds for  $K-1$ . To do this we employ (16) and (17).

First of all, (16) and (17) imply that  $\nu_{X,K-1}$  is absolutely continuous with respect to Lebesgue measure on  $\mathcal{X}(K-1)$  and has the density

$$(\nu_{X,K} \Lambda_{K-1}^K)(B) = \int_{A \in \mathcal{X}(K)} \nu_{X,K}(dA) \Lambda_{K-1}^K(A, B), \quad B \in \mathcal{X}(K-1). \quad (18)$$

Let us compute the integral explicitly. By the induction assumption,  $\nu_{X,K}$  is absolutely continuous and has density (15). Therefore, integral (18) can be written in the form

$$\int_{A \in \mathcal{X}^0(K)} M(a_1, \dots, a_K; x_1, \dots, x_N) \Lambda_{K-1}^K(A, B) da_1 \dots da_K.$$

Write  $B = (b_1, \dots, b_K)$ . Substituting the explicit expression for  $\Lambda_{K-1}^K(A, B)$  given by Proposition 3.1 we rewrite this as

$$(K-1)!V(B) \int_A \frac{M(a_1, \dots, a_K; x_1, \dots, x_N)}{V(A)} da_1 \dots da_K, \quad (19)$$

where the integration domain is

$$-\infty < a_1 \leq b_1, \quad \dots, \quad b_i \leq a_{i+1} \leq b_{i+1}, \quad \dots, \quad b_{K-1} \leq a_K < +\infty. \quad (20)$$

Next, plug in into (19) the explicit expression for  $M(a_1, \dots, a_K; x_1, \dots, x_N)$  given by (15). Then the factor  $V(A)$  is cancelled out and we get

$$\frac{c_{N,K}(K-1)!V(B)}{\prod_{(j,i): j-i \geq N-K+1} (x_j - x_i)} \int_A \det [M(a_j; x_i, \dots, x_{N-K+i})]_{i,j=1}^K da_1 \dots da_K \quad (21)$$

with the same integration domain (20).

Put aside the pre-integral factor in (21) and examine the integral itself. It can be written as a  $K \times K$  determinant,

$$\det[F(i, j)]_{i, j=1}^K,$$

where

$$F(i, j) := \int_{b_{j-1}}^{b_j} M(a; Y_i) da$$

and

$$Y_i := (x_i, \dots, x_{N-K+i})$$

with the understanding that  $b_0 = -\infty$  and  $b_K = +\infty$ .

We are going to prove that

$$\begin{aligned} \det[F(i, j)]_{i, j=1}^K &= (N - K + 1)^{K-1} \prod_{i=1}^{K-1} (x_{N-K+i+1} - x_i) \\ &\quad \times \det[M(b_j; x_i, \dots, x_{N-K+i+1})]_{i, j=1}^{K-1}. \end{aligned} \quad (22)$$

This will justify the induction step, because

$$c_{N, K} = c_{N, K-1} \cdot \frac{(N - K + 1)^{K-1}}{(K - 1)!}$$

and

$$\frac{\prod_{i=1}^{K-1} (x_{N-K+i+1} - x_i)}{\prod_{(j, i): j-i \geq N-K+1} (x_j - x_i)} = \frac{1}{\prod_{(j, i): j-i \geq N-K+2} (x_j - x_i)}.$$

*Step 4.* It remains to prove (22). We evaluate the quantities  $F(i, j)$  using Lemma 2.2, where we substitute  $n = N - K + 1$  and  $Y = Y_i$ . Then we get that the matrix entries  $F(i, j)$  are given by the following formulas:

- The entries of the first column have the form

$$F(i, 1) = 1 - f_{b_1}[Y_i] \quad \text{by (6).}$$

- The entries of the  $j$ th column,  $2 \leq j \leq K - 1$ , have the form

$$F(i, j) = f_{b_{j-1}}[Y_i] - f_{b_j}[Y_i] \quad \text{by (7).}$$

- The entries of the last column have the form

$$F(i, K) = f_{b_K}[Y_i] \quad \text{by (8).}$$

We have  $\det F = \det G$ , where the  $K \times K$  matrix  $G$  is defined by

$$G(i, j) := F(i, j) + \dots + F(i, K).$$

The entries of the matrix  $G$  are

$$G(i, 1) = 1, \quad G(i, j) = f_{b_{j-1}}[Y_i], \quad 2 \leq j \leq K.$$

Next, we get  $\det G = \det H$  with the  $(K-1) \times (K-1)$  matrix  $H$  defined by

$$H(i, j) := F(i+1, j+1) - F(i, j), \quad 1 \leq i, j \leq K-1.$$

Observe now that

$$H(i, j) = f_{b_j}[Y_{i+1}] - f_{b_j}[Y_i],$$

which can be rewritten as

$$H(i, j) = (x_{N-K+i+1} - x_i) \frac{f_{b_j}[x_{i+1}, \dots, x_{N-K+i+1}] - f_{b_j}[x_i, \dots, x_{N-K+i}]}{x_{N-K+i+1} - x_i} \quad (23)$$

$$= (x_{N-K+i+1} - x_i) f_{b_j}[x_i, \dots, x_{N-K+i+1}] \quad \text{by (3)} \quad (24)$$

$$= \frac{1}{N-K+1} (x_{N-K+i+1} - x_i) M(b_j; x_i, \dots, x_{N-K+i+1}) \quad \text{by (4)}. \quad (25)$$

This shows that the determinant  $\det H = \det[H(i, j)]_{i,j=1}^{K-1}$  equals the right-hand side of (22). Since  $\det H = \det F$ , this completes the proof.  $\square$

#### 4. ACKNOWLEDGEMENT

I am grateful to Jacques Faraut for valuable comments. The work was partially supported by a grant from Simons Foundation (Simons–IUM Fellowship) and the project SFB 701 of Bielefeld University.

#### REFERENCES

- [1] Yu. Baryshnikov, *GUEs and queues*. Prob. Theory Rel. Fields **119** (2001), 256–274.
- [2] C. de Boor, *A practical guide to splines*. Springer, 1978.
- [3] A. Borodin, G. Olshanski, *The boundary of the Gelfand-Tsetlin graph: A new approach*. Adv. Math. **230** (2012), 1738–1779; arXiv:1109.1412.
- [4] H. B. Curry and I. J. Schoenberg, *On Polya frequency functions IV: The fundamental spline functions and their limits*. J. Analyse Math. **17** (1966), 71–107.
- [5] P. J. Davis, *Interpolation and approximation*. Dover, 1975.
- [6] M. Defosseux, *Orbit measures, random matrix theory and interlaced determinantal processes*. Ann. Inst. Henri Poincaré (B) Prob. Stat. **46** (2010), 209–249; arXiv:0810.1011.
- [7] J. Faraut, *Noyau de Peano et intégrales orbitales*. Glob. J. Pure Appl. Math. **1** (2005), no. 3, 306–320.
- [8] I. M. Gelfand, M. A. Naimark, *Unitary representations of classical groups*. Proc. Steklov Math. Institute, vol. 36 (1950) (Russian); German translation: I. M. Gelfand, M. A. Neumark, *Unitäre Darstellungen der klassischen Gruppen*. Mathematische Lehrbücher und Monographien. II. Abt. Band 6. Berlin: Akademie-Verlag (1957).
- [9] Yu. A. Neretin, *Rayleigh triangles and nonmatrix interpolation of matrix beta integrals*. Mat. Sb. **194** (2003), no. 4, 49–74 (Russian); English translation in Sbornik: Mathematics **194** (2003), no. 3–4, 515–540; arXiv:math/0301070.
- [10] G. Olshanski, A. Vershik, *Ergodic unitarily invariant measures on the space of infinite Hermitian matrices*. In: Contemporary Mathematical Physics. F. A. Berezin’s memorial volume.

- American Mathematical Society Translations, Series 2, Vol. 175 (Advances in the Mathematical Sciences — 31), R. L. Dobrushin, R. A. Minlos, M. A. Shubin, A. M. Vershik, eds., Amer. Math. Soc., Providence, RI, 1996, pp. 137–175; arXiv:math/9601215.
- [11] L. Petrov, *The boundary of the Gelfand-Tsetlin graph: New proof of Borodin-Olshanski's formula, and its  $q$ -analogue*. Moscow Math. J., to appear; arXiv:1208.3443.
  - [12] G. M. Phillips, *Approximation and interpolation by polynomials*. CMS Books in Math. vol. 14. Springer, 2003.
  - [13] H. Weyl, *The classical groups. Their invariants and representations*. Princeton Univ. Press, 1939; 1997 (fifth edition).

INSTITUTE FOR INFORMATION TRANSMISSION PROBLEMS, MOSCOW, RUSSIA;  
INDEPENDENT UNIVERSITY OF MOSCOW, RUSSIA;  
NATIONAL RESEARCH UNIVERSITY HIGHER SCHOOL OF ECONOMICS, MOSCOW, RUSSIA  
*E-mail address:* olsh2007@gmail.com